

A GENERALIZED SPECTRAL RADIUS FORMULA AND OLSEN'S QUESTION

TERRY LORING AND TATIANA SHULMAN

ABSTRACT. Let A be a C^* -algebra and I be a closed ideal in A . For $x \in A$, its image under the canonical surjection $A \rightarrow A/I$ is denoted by \dot{x} , and the spectral radius of x is denoted by $r(x)$. We prove that

$$\max\{r(x), \|\dot{x}\|\} = \inf \|(1+i)^{-1}x(1+i)\|$$

(where infimum is taken over all $i \in I$ such that $1+i$ is invertible), which generalizes spectral radius formula of Murphy and West [5] (Rota for $\mathcal{B}(\mathcal{H})$ [4]). Moreover if $r(x) < \|\dot{x}\|$ then the infimum is attained. A similar result is proved for commuting family of elements of a C^* -algebra.

Using this we give a partial answer to an open question of C. Olsen: if p is a polynomial then for "almost every" operator $T \in B(H)$ there is a compact perturbation $T + K$ of T such that

$$\|p(T + K)\| = \|p(T)\|_e.$$

We show also that if operators A, B commute, A is similar to a contraction and B is similar to a strict contraction then they are simultaneously similar to contractions.

1. INTRODUCTION

Let A be a C^* -algebra, x its element. The spectral radius of x will be denoted by $r(x)$. Spectral radius formula of Murphy and West [5] (Rota for $\mathcal{B}(\mathcal{H})$ [4]) is

$$r(x) = \inf \|s^{-1}xs\|,$$

where infimum is taken over all invertible elements in the C^* -algebra. In present paper we get the following generalization of this formula. Let I be a closed ideal in A . For $x \in A$, by \dot{x} we denote the image of x under the canonical surjection $A \rightarrow A/I$. We prove (Corollary 3) that

$$\max\{r(x), \|\dot{x}\|\} = \inf \|(1+i)^{-1}x(1+i)\|,$$

where infimum is taken over all $i \in I$ such that $1+i$ is invertible. In the case $I = A$ it is the spectral radius formula. Moreover we prove that if $\|\dot{x}\| > r(x)$ then the infimum is attained. For a finite family of commuting elements, the corresponding i can be chosen common (Theorem 1).

We apply these results to a question posed by C. Olsen in [9]. Her question is the following:

given an operator T and a polynomial p does there exist a compact perturbation $T + K$ of T such that $\|p(T + K)\| = \|p(T)\|_e$?

(here $\|\cdot\|_e$ is the essential norm of an operator). Olsen's second question was if this compact operator K can be chosen simultaneously for all polynomials.

The both questions are open, but there are partial results ([12], [7], [8], [9], [14], [15], [13]), obtained either for special classes of operators or for special polynomials. Below we list most of these results:

1) If p is arbitrary and T is essentially normal or subnormal or n -normal operator or a nilpotent weighted shift, then there is a positive answer to Olsen's question. Moreover K can be chosen independently of polynomial [8], [9];

2) If $p(T)$ is compact, then the answer is positive [7];

3) If T is arbitrary, p is linear, the answer is positive. Moreover K can be chosen common for all linear polynomials [14];

4) If p is a monomial $p(x) = x^n$, T arbitrary - the answer is positive. Moreover K can be chosen common for finitely many monomials. If T is not quasinilpotent, then K can be chosen common for all monomials [12].

In present paper for finitely many arbitrary polynomials p_1, \dots, p_n we show (Theorems 6 and 10) that there is a dense open subset $\Sigma_{p_1, \dots, p_n} \subset \mathcal{B}(\mathcal{H})$ such that for any operator $T \in \Sigma_{p_1, \dots, p_n}$, there is $K \in \mathcal{K}(\mathcal{H})$ such that

$$\|p_i(T + K)\| = \|p_i(T)\|_e,$$

$i = 1, \dots, n$.

In other words, those operators for which we cannot answer Olsen's question, form a nowhere dense set.

In [8], Theorem 5.2, for either T or T^* quasitriangular there was solved an "approximate version" of the problem:

$$\inf_{K \in \mathcal{K}(\mathcal{H})} \|p(T + K)\| = \|p(T)\|_e.$$

We prove that this holds for arbitrary operator T (Theorem 15).

Finally, in Section 3, we present a result which is not formally related with generalized spectral radius formula, but uses arguments similar to ones used in the proof of the formula. It is an open question whether two commuting operators which are each similar to a contraction are simultaneously similar to contractions ([17], page 159). In [18] a confirmative answer is obtained for the case when the both operators are similar to strict contractions. We show (Theorem 17) that if only one of these operators is similar to a strict contraction, then the answer is positive.

The authors are grateful to Victor Shulman for many helpful discussions.

2. GENERALIZED SPECTRAL RADIUS FORMULA

Theorem 1. *Let A be a C^* -algebra, I its ideal, and $a_1, \dots, a_n \in A$ commute. Then for any $\epsilon > 0$, there is $e \in I$ such that*

$$\|(1 + e)a_j(1 + e)^{-1}\| - \epsilon \leq \max\{r(a_j), \|\dot{a}_j\|\} \leq \|(1 + e)a_j(1 + e)^{-1}\|,$$

for all j . If $r(a_j) < \|\dot{a}_j\|$, for all j , then there is $e \in I$ such that

$$\max\{r(a_j), \|\dot{a}_j\|\} = \|(1 + e)a_j(1 + e)^{-1}\|,$$

for all j .

We will need a lemma.

Lemma 2. *Let A be a C^* -algebra, I its ideal, and $a_1, \dots, a_n \in A$ commute. Suppose $r(a_j) < 1$, $\|\dot{a}_j\| \leq 1$, for all i . Then there exists $e \in I$ such that*

$$\|(1 + e)a_j(1 + e)^{-1}\| \leq 1,$$

for all j .

Proof. By [16] there exists $0 \leq i_0 \leq 1$ in I such that $\|(1 - i_0)a_j\| \leq 1$, $j = 1, \dots, n$. Let $i = 1 - (1 - i_0)^2$. Then

$$a_j^*(1 - i)a_j \leq 1,$$

$j = 1, \dots, n$.

Since $r(a_j) < 1$, by Cauchy's root test

$$\sum_{k=1}^{\infty} \|a_j^k\|^2 < \infty. \quad (1)$$

Consider series

$$\sum_{k_1, \dots, k_n} (a_1^*)^{k_1} \dots (a_n^*)^{k_n} i a_n^{k_n} \dots a_1^{k_1}.$$

For its partial sums we have

$$\left\| \sum_{k_1, \dots, k_n=1}^N (a_1^*)^{k_1} \dots (a_n^*)^{k_n} i a_n^{k_n} \dots a_1^{k_1} \right\| \leq \sum_{k_1, \dots, k_n=1}^N \|a_1^{k_1}\|^2 \dots \|a_n^{k_n}\|^2$$

and by (1) the series converges.

Let

$$z = 1 + \sum_{k_1 + \dots + k_n \geq 1} (a_1^*)^{k_1} \dots (a_n^*)^{k_n} i a_n^{k_n} \dots a_1^{k_1}.$$

Then $z \geq 1$ and hence invertible. Let $y = z^{1/2}$. Then y is invertible and of the form $1 + e$, $e \in I$. For any $1 \leq j \leq n$ we have

$$\begin{aligned} a_j^* y^2 a_j &= a_j^* (1 - i) a_j + \sum_{k_1 + \dots + k_n \geq 1} (a_1^*)^{k_1} \dots (a_j^*)^{k_j+1} \dots (a_n^*)^{k_n} i a_n^{k_n} \dots a_j^{k_j+1} \dots a_1^{k_1} \\ &\leq 1 + \sum_{k_1 + \dots + k_n \geq 1} (a_1^*)^{k_1} \dots (a_n^*)^{k_n} i a_n^{k_n} \dots a_1^{k_1} = y^2 \end{aligned}$$

and

$$\|y a_j y^{-1}\|^2 = \|y^{-1} a_j^* y^2 a_j y^{-1}\| \leq 1.$$

□

Proof of Theorem 1. For each $j = 1, \dots, n$ take any δ_j such that

$$\max\{r(a_j), \|\dot{a}_j\|\} < \delta_j \leq \max\{r(a_j), \|\dot{a}_j\|\} + \epsilon.$$

Applying Lemma 2 to the elements $\frac{a_j}{\delta_j}$, we find $e \in I$ such that

$$\|(1 + e) \frac{a_j}{\delta_j} (1 + e)^{-1}\| \leq 1,$$

$j = 1, \dots, n$. Hence

$$\|(1 + e) a_j (1 + e)^{-1}\| \leq \delta_j \leq \epsilon + \max\{r(a_j), \|\dot{a}_j\|\}$$

$j = 1, \dots, n$. The inequality

$$\|(1 + e) a_j (1 + e)^{-1}\| \geq \max\{r(a_j), \|\dot{a}_j\|\}$$

is clear.

If $\|\dot{a}_j\| > r(a_j)$, we may take $\delta_j = \|\dot{a}_j\|$ and obtain the second statement. □

In particular, for a single element we get the following formula.

Corollary 3. *Let A be a C^* -algebra, I its ideal, $a \in A$. Then*

$$\max\{r(a), \|\dot{a}\|\} = \inf \|(1+e)a(1+e)^{-1}\|$$

(here infimum in the right-hand side is taken over all $e \in I$ such that $1+e$ is invertible). If $\|\dot{a}\| > r(a)$ then the infimum is attained.

Remark 4. *In the case $r(a) \geq \|\dot{a}\|$, infimum in the formula need not be attained. First of all, if $r(a) = \|\dot{a}\| = 0$, it obviously is not attained.*

If $r(a) \geq \|\dot{a}\| \neq 0$ it also might not be attained. For example, consider an operator of the form

$$T = \begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix},$$

where K is compact. Then $r(T) = \|T\|_e = 1$. If infimum in the formula was attained, T would be similar to a contraction and hence power-bounded. But

$$\|T^n\| = \left\| \begin{pmatrix} 1 & nK \\ 0 & 1 \end{pmatrix} \right\| \rightarrow \infty.$$

3. OLSEN'S QUESTION

Let $\mathcal{B}(\mathcal{H})$ be the space of all bounded operators on a Hilbert space \mathcal{H} , $\mathcal{K}(\mathcal{H})$ the ideal of all compact operators, $Q(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ Calkin algebra, $q : \mathcal{B}(\mathcal{H}) \rightarrow Q(\mathcal{H})$ the canonical surjection.

For $T \in \mathcal{B}(\mathcal{H})$, $\|\dot{T}\|$ and $r(\dot{T})$ are called the essential norm and the essential spectral radius of T . We will use also their standard denotations $\|T\|_e$ and $r_e(T)$.

Let p_1, \dots, p_n be polynomials. Define

$$G_{p_1, \dots, p_n} = \{a \in Q(\mathcal{H}) \mid r(p_i(a)) < \|p_i(a)\|, i = 1, \dots, n\}$$

and

$$\Sigma_{p_1, \dots, p_n} = q^{-1}(G_{p_1, \dots, p_n}).$$

Lemma 5. *For any polynomials p_1, \dots, p_n ,*

$$\Sigma_{p_1, \dots, p_n} = \{T \in \mathcal{B}(\mathcal{H}) \mid \exists K \in \mathcal{K}(\mathcal{H}) \text{ such that } r(p_i(T+K)) < \|p_i(T)\|_e, i = 1, \dots, n\}.$$

Proof. Suppose for $T \in \mathcal{B}(\mathcal{H})$ there exists $K \in \mathcal{K}(\mathcal{H})$ such that $r(p_i(T+K)) < \|p_i(T)\|_e$, for all i . Then

$$r(p_i(\dot{T})) \leq r(p_i(T+K)) < \|p_i(T)\|_e,$$

whence $\dot{T} \in G_{p_1, \dots, p_n}$ and $T \in \Sigma_{p_1, \dots, p_n}$.

Suppose $T \in \Sigma_{p_1, \dots, p_n}$. By [2] there is $K \in \mathcal{K}(\mathcal{H})$ such that $\sigma(T+K)$ is obtained by filling in some holes in the essential spectrum of T . Then by the maximum principle

$$r(p_i(T+K)) = \max_{t \in \partial\sigma(T+K)} |p_i(t)| = \max_{t \in \partial\sigma_e(T)} |p_i(t)| = r(p_i(\dot{T})) < \|p_i(T)\|_e,$$

$i = 1, \dots, n$. □

Theorem 6. *For any polynomials p_1, \dots, p_n and an operator $T \in \Sigma_{p_1, \dots, p_n}$, Olsen's question has a positive answer. That is there is a compact operator K such that $\|p_i(T+K)\| = \|p_i(T)\|_e$, $i = 1, \dots, n$.*

Proof. By Lemma 5 there is $K_1 \in \mathcal{K}(\mathcal{H})$ such that $r(p_i(T + K_1)) < \|p_i(T)\|_e$, $i = 1, \dots, n$. By Theorem 1 there exists $K_2 \in \mathcal{K}(\mathcal{H})$ such that for all i

$$\|p_i(T)\|_e = \|(1 + K_2)^{-1}p_i(T + K_1)(1 + K_2)\| = \|p_i((1 + K_2)^{-1}(T + K_1)(1 + K_2))\|.$$

Let $K = (1 + K_2)^{-1}(T + K_1)(1 + K_2) - T$. Then $K \in \mathcal{K}(\mathcal{H})$ and $\|p_i(T + K)\| = \|p_i(T)\|_e$ for all i . \square

Below we present some new examples of operators for which Olsen's question has a positive answer.

3.1. Special cases.

Theorem 7. *For any quasinilpotent operator and any polynomial p such that $p(0) = 0$, Olsen's question has a positive answer.*

Proof. Let T be a quasinilpotent operator. If $p(T)$ is compact, the assertion follows from Olsen's structure theorem for polynomially compact operators ([7]). If $p(T)$ is not compact, then $r(p(T)) < \|p(T)\|_e$ and $T \in \Sigma_p$. The assertion follows now from Theorem 6. \square

For nilpotent operators, one does not even need an assumption $p(0) = 0$.

Lemma 8. *Let $T \in \mathcal{B}(\mathcal{H})$ and $(T - t_N)^{k_N}(T - t_{N-1})^{k_{N-1}} \dots (T - t_1)^{k_1} = 0$. Let*

$$\begin{aligned} \mathcal{H}_1 &= \text{Ker}(T - t_1) \\ \mathcal{H}_2 &= \text{Ker}(T - t_1)^2 \ominus \mathcal{H}_1 \\ &\dots \\ \mathcal{H}_{k_1} &= \text{Ker}(T - t_1)^{k_1} \ominus \mathcal{H}_{k_1-1} \\ \mathcal{H}_{k_1+1} &= \text{Ker}(T - t_2)(T - t_1)^{k_1} \ominus \mathcal{H}_{k_1} \\ &\dots \\ \mathcal{H}_{k_1+\dots+k_N-1} &= \text{Ker}(T - t_N)^{k_N-1}(T - t_{N-1})^{k_{N-1}} \dots (T - t_1)^{k_1} \ominus \mathcal{H}_{k_1+\dots+k_N-2}. \end{aligned}$$

Then with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_{k_1+\dots+k_N-1}$ the operator T has uppertriangular form with $t_1 1, \dots, t_1 1, \dots, t_N 1, \dots, t_N 1$ on the diagonal, where each $t_i 1$ is repeated k_i times.

Proof. If $x \in \mathcal{H}_1$, then $Tx = t_1 x$. If $x \in \mathcal{H}_2$, then $Tx = (T - t_1)x + t_1 x$, where $(T - t_1)x \in \mathcal{H}_1$. And so on. \square

Theorem 9. *For any nilpotent operator T and any polynomial p , Olsen's question has a positive answer.*

Proof. Let n be such that $T^n = 0$. By Lemma 8, T is an n -block upper-triangular operator with zeros at the diagonal. Hence $p(T)$ is n -block upper-triangular with $p(0)$ at the diagonal. Hence

$$r(p(T)) = |p(0)|.$$

If $\|p(T)\|_e > r(p(T))$, then by Theorem 6, we are done. So let us assume

$$\|p(T)\|_e = r(p(T)) = |p(0)|.$$

The image $q(p(T))$ of $p(T)$ in Calkin algebra is also n -block upper-triangular operator with entries in Calkin algebra and with $p(0)$ at the diagonal. Its norm can be equal $|p(0)|$ only if it is diagonal. Thus we have

$$p(T) - p(0)1 \in \mathcal{K}(\mathcal{H}).$$

Let $\tilde{p} = p - p(0)$. Then $\tilde{p}(T) \in \mathcal{K}(\mathcal{H})$ and by the Olsen's structure theorem for polynomially compact operators ([7]), there exists $K \in \mathcal{K}(\mathcal{H})$ such that

$$\begin{aligned}\tilde{p}(T + K) &= 0, \\ p(T + K) &= p(0)1, \\ \|p(T + K)\| &= |p(0)| = \|p(T)\|_e.\end{aligned}$$

□

Our purpose now is to show that for any polynomials p_1, \dots, p_n , the set Σ_{p_1, \dots, p_n} consists of almost all operators.

3.2. The set Σ_{p_1, \dots, p_n} .

Theorem 10. *The complement of Σ_{p_1, \dots, p_n} is nowhere dense.*

We will need several lemmas.

For any set $S \subseteq \mathbb{C}$, we denote its boundary by ∂S .

Lemma 11. *Let $T \in \mathcal{B}(\mathcal{H})$ and let p be a polynomial. Then there is $\lambda_0 \in \sigma_e(T)$ such that $T - \lambda_0$ is not semi-Fredholm operator and $|p(\lambda_0)| = r_e(p(T))$.*

Proof. By [2] there exists $K \in \mathcal{K}(\mathcal{H})$ such that $\sigma(T + K)$ is obtained from $\sigma_e(T)$ by filling in some holes. Hence

$$\partial \sigma_e(T) \supseteq \partial \sigma(T + K). \quad (2)$$

By the maximum principle

$$r_e(p(T)) = \max_{t \in \partial \sigma_e(T)} |p(t)| \geq \max_{t \in \partial \sigma(T + K)} |p(t)| = r(p(T + K)).$$

The opposite inequality is obvious. Thus

$$r_e(p(T)) = r(p(T + K)). \quad (3)$$

By the maximum principle there is a point $\lambda_0 \in \partial \sigma(T + K)$ such that

$$|p(\lambda_0)| = r(p(T + K)).$$

By (3) and (2) $\lambda_0 \in \sigma_e(T)$ and $|p(\lambda_0)| = r_e(p(T))$.

Suppose $T - \lambda_0 1$ is semi-Fredholm. Then all operators in some neighborhood of $T - \lambda_0 1$ are semi-Fredholm of the same index [3]. Since $\lambda_0 \in \partial \sigma(T + K)$, in this neighborhood there is an operator $T - \lambda 1$ such that $T + K - \lambda 1$ is invertible. Hence

$$\text{ind}(T - \lambda_0 1) = \text{ind}(T - \lambda 1) = \text{ind}(T + K - \lambda 1) = 0.$$

Thus $T - \lambda_0 1$ is semi-Fredholm operator of index 0, which implies that it is Fredholm. Since $\lambda_0 \in \sigma_e(T)$, it is a contradiction. □

The following lemma is inspired by work of Holmes, Scranton and Ward ([10]).

Lemma 12. *Let $T \in \mathcal{B}(\mathcal{H})$, p be a polynomial. Then for any $\epsilon > 0$ there is $A \in \mathcal{B}(\mathcal{H})$ such that*

$$\|T - A\| \leq \epsilon, \quad \|p(A)\|_e > r_e(p(A)).$$

Proof. Let λ_0 be as in Lemma 11. Since $T - \lambda_0 1$ is not semi-Fredholm, there is an infinite-dimensional projection Q , such that $(T - \lambda_0 1)Q \in \mathcal{K}(\mathcal{H})$ ([1]). Let $\mathcal{L} = Q\mathcal{H}$, $\mathcal{N} = \mathcal{L}^\perp$. With respect to the decomposition $\mathcal{H} = \mathcal{L} \oplus \mathcal{N}$, we write

$$T = \begin{pmatrix} \lambda_0 1 + K_{11} & * \\ K_{21} & T_1 \end{pmatrix},$$

where $K_{11}, K_{21} \in \mathcal{K}(\mathcal{H})$. For any operator $S : \mathcal{L} \rightarrow \mathcal{L}$ we have

$$\begin{aligned} T + SQ &= \begin{pmatrix} \lambda_0 1 + S + K_{11} & * \\ K_{21} & T_1 \end{pmatrix}, \\ p(T + SQ) &= \begin{pmatrix} p(\lambda_0 1 + S) + K'_{11} & * \\ K'_{21} & p(T_1) + K \end{pmatrix}, \end{aligned} \quad (4)$$

where $K'_{11}, K'_{21}, K \in \mathcal{K}(\mathcal{H})$. Hence

$$\sigma_e(p(T + SQ)) = \sigma_e \left(\begin{pmatrix} p(\lambda_0 1 + S) & * \\ 0 & p(T_1) \end{pmatrix} \right) = \sigma_e(p(\lambda_0 1 + S)) \cup \sigma_e(p(T_1)).$$

If S is quasinilpotent, then

$$\sigma_e(p(\lambda_0 1 + S)) = \sigma_e(p(\lambda_0)1).$$

Hence

$$\sigma_e(p(T + SQ)) = \sigma_e(p(T)). \quad (5)$$

By (5) and the choice of λ_0 ,

$$r_e(p(T + SQ)) = r_e(p(T)) = |p(\lambda_0)|. \quad (6)$$

So it suffices to construct a nilpotent operator S such that $\|p(T + SQ)\| > |p(\lambda_0)|$ and $\|S\| \leq \epsilon$. Then for $A = T + SQ$ we would have $\|T - A\| \leq \epsilon$ and $\|p(A)\|_e > r_e(p(A))$.

Write p in the form $p(t) = a_0 + a_1 t + \dots + a_n t^n$, where $a_n \neq 0$. Then for any operator S ,

$$p(\lambda_0 + S) = a_0 1 + \mu_1 S + \dots + \mu_n S^n,$$

where

$$\begin{aligned} \mu_1 &= a_1 + a_2 C_2^1 \lambda_0 + \dots + a_n C_n^1 \lambda_0^{n-1}, \\ \mu_2 &= a_2 + a_3 C_3^2 \lambda_0 + \dots + a_n C_n^2 \lambda_0^{n-1}, \\ &\dots \quad \mu_n = a_n, \end{aligned}$$

and $C_n^k = \frac{n!}{(n-k)!k!}$. Choose the smallest i such that $\mu_1 = 0, \dots, \mu_{i-1} = 0, \mu_i \neq 0$. It exists because $\mu_n \neq 0$. Let S be a block-diagonal operator

$$S = \begin{pmatrix} S' & & \\ & S' & \\ & & \ddots \end{pmatrix},$$

where S' is $(i+1)$ by $(i+1)$ nilpotent matrix

$$S' = \begin{pmatrix} 0 & & & & \\ \epsilon & 0 & & & \\ & \epsilon & 0 & & \\ & & \ddots & \ddots & \\ & & & \epsilon & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \|S\| &= \epsilon, \\ S^{i+1} &= 0, \quad p(\lambda_0 + S) = p(\lambda_0)1 + \mu_i S^i = \begin{pmatrix} p(\lambda_0)1 + \mu_i S^i & & \\ & p(\lambda_0)1 + \mu_i S^i & \\ & & \ddots \end{pmatrix}, \end{aligned}$$

and

$$\|p(\lambda_0 + S)\|_e = \|p(\lambda_0)1 + \mu_i S^i\| > |p(\lambda_0)|. \quad (7)$$

By (4), (6), (7) we get

$$\|p(T + SQ)\|_e = \left\| \begin{pmatrix} p(\lambda_0 1 + S) & * \\ 0 & p(T_1) \end{pmatrix} \right\|_e \geq \|p(\lambda_0 1 + S)\|_e > |p(\lambda_0)|.$$

□

Corollary 13. G_p is dense in $Q(\mathcal{H})$.

Lemma 14. G_p is open in $Q(\mathcal{H})$.

Proof. Let a_n belong to the complement of G_p , $a_n \rightarrow a$. Then $p(a_n) \rightarrow p(A)$. By the semi-continuity of spectral radius,

$$r(p(a)) \geq \limsup r(p(a_n)) = \limsup \|p(a_n)\| = \|p(a)\|.$$

The opposite inequality is obvious. Thus the complement of G_p is closed. □

Proof of Theorem 10. Lemma 14 implies that Σ_{p_i} is open in $\mathcal{B}(\mathcal{H})$, for each i . Let us prove that each Σ_{p_i} is dense. Let $B \subset \mathcal{B}(\mathcal{H})$ be an open set. Since q is an open map, it follows from Corollary 13 that $q(B) \cap G_{p_i} \neq \emptyset$. Therefore there exist $T \in B$, $A \in \Sigma_{p_i}$ such that $q(T) = q(A)$, whence $T - A \in \mathcal{K}(\mathcal{H})$. Since Σ_{p_i} is closed under compact perturbations, $T \in \Sigma_{p_i}$ and $B \cap \Sigma_{p_i} \neq \emptyset$. Thus each Σ_{p_i} is open and dense and since intersection of finitely many open dense sets is open and dense, we conclude that

$$\Sigma_{p_1, \dots, p_n} = \bigcap_{i=1}^n \Sigma_{p_i}$$

is open and dense. This implies that its complement is nowhere dense. □

Finally we get an approximate version of Olsen's question.

Theorem 15. Let $T \in \mathcal{B}(\mathcal{H})$. There exists a sequence $K_n \in \mathcal{K}(\mathcal{H})$ such that for each polynomial p ,

$$\|p(T + K_n)\| \rightarrow \|p(T)\|_e.$$

Proof. Arguing as in the proof of Lemma 11, one find a compact perturbation $T + K$ of T , such that for each polynomial p ,

$$r(p(T + K)) = r_e(p(T + K)) \leq \|p(T + K)\|_e. \quad (8)$$

Let $\mathcal{F} = \{p_1, p_2, \dots\}$ be the set of all polynomials with rational coefficients. Applying Theorem 1 to the family $\{p_j(T + K)\}$, $j = 1, \dots, n$, and using (8), we find $e_n \in \mathcal{K}(\mathcal{H})$ such that

$$\|(1 + e_n)p_j(T + K)(1 + e_n)^{-1}\| \leq \|p_j(T + K)\|_e + \frac{1}{n},$$

$j = 1, \dots, n$ and hence

$$\|p_j((1 + e_n)(T + K)(1 + e_n)^{-1})\| \leq \|p_j(T + K)\|_e + \frac{1}{n},$$

$j = 1, \dots, n$. Let

$$K_n = (1 + e_n)(T + K)(1 + e_n)^{-1} - T.$$

Then $K_n \in \mathcal{K}(\mathcal{H})$. Since for any polynomial p , $\|p(T + K)\|_e = \|p(T)\|_e$, we have

$$\|p_j(T + K_n)\| \leq \|p_j(T)\|_e + \frac{1}{n}$$

for $j = 1, \dots, n$.

Hence for any $p \in \mathcal{F}$, there is n_p such that for all $n > n_p$,

$$\|p(T + K_n)\| \leq \|p(T)\|_e + \frac{1}{n}, \quad (9)$$

and applying this to the polynomial $p(z) = z$, we conclude that the sequence K_n is bounded. Let $C = \sup \|T + K_n\|$. For arbitrary polynomial p there is $p_{k_n} \in \mathcal{F}$ such that

$$\sup_{|z| \leq C} |p(z) - p_{k_n}(z)| < \frac{1}{n}.$$

Now using (9) and von Neumann's inequality, we get for all sufficiently large n

$$\begin{aligned} \|p(T + K_n)\| &\leq \|p_{k_n}(T + K_n)\| + \|(p - p_{k_n})(T + K_n)\| \leq \\ &\|p_{k_n}(T + K_n)\| + \sup_{|z| \leq C} |p_{k_n}(z) - p(z)| < \|p_{k_n}(T)\|_e + \frac{2}{n} \leq \\ &\|p(T)\|_e + \sup_{|z| \leq \|T\|_e} |p_{k_n}(z) - p(z)| + \frac{2}{n} \leq \|p(T)\|_e + \frac{3}{n}. \end{aligned}$$

Since $\|p(T + K_n)\| \geq \|p(T)\|_e$, we obtain

$$\|p(T + K_n)\| \rightarrow \|p(T)\|_e.$$

□

4. COMMON SIMILARITY TO CONTRACTIONS

Lemma 16. *Let A be a C^* -algebra, $a, b \in A$, $[a, b] = 0$, $r(a) < 1$, $\|b\| \leq 1$. Then there is an invertible $y \in A$ such that*

$$\|yay^{-1}\| < 1, \quad \|yby^{-1}\| \leq 1.$$

Proof. Let $z = 1 + a^*a + (a^*)^2a^2 + \dots$ and let $y = z^{1/2}$. By the proof of Lemma 2, applied to the case of $I = A$ and a single element a ,

$$\|yay^{-1}\| < 1.$$

We have

$$\begin{aligned} b^*y^2b &= b^*b + b^*a^*ab + b^*(a^*)^2a^2b + \dots = b^*b + a^*b^*ba + (a^*)^2b^*ba^2 + \dots \leq \\ &b^*b + a^*a + (a^*)^2a^2 + \dots = b^*b - 1 + y^2 \leq y^2 \end{aligned}$$

and

$$\|yby^{-1}\|^2 = \|y^{-1}b^*y^2by^{-1}\| \leq 1.$$

□

Theorem 17. *Let A be a C^* -algebra, $a, b \in A$, $[a, b] = 0$, a is similar to a strict contraction and b is similar to a contraction. Then there is an invertible $c \in A$ such that*

$$\|cac^{-1}\| \leq 1, \quad \|cbc^{-1}\| \leq 1.$$

Proof. Since b is similar to a contraction, there is s such that

$$\|sbs^{-1}\| \leq 1.$$

Since a is similar to a strict contraction,

$$r(sas^{-1}) = r(a) < 1.$$

Also we have

$$[sas^{-1}, sbs^{-1}] = 0.$$

By Lemma 16, there is y such that

$$\|ysas^{-1}y^{-1}\| < 1, \quad \|ysbs^{-1}y^{-1}\| \leq 1.$$

□

REFERENCES

- [1] F. Wolf, On the invariance of the essential spectrum under a change of boundary conditions of partial differential boundary operators, *Indag. Math.* 21 (1959), 142-147
- [2] J. G. Stampfli, Compact perturbations, normal eigenvalues and a problem of Salinas. *J. London Math. Soc.* (2), 9 (1974)
- [3] T. Kato, *Perturbation Theory for Linear Operators*, Springer, New-York, 1966.
- [4] G. Rota, On models for linear operators, *Comm. Pure Appl. Math.* 13 (1960), 469-472.
- [5] Murphy, G. J., West, T. T., Spectral radius formulae. *Proc. Edinburgh Math. Soc.* (2) 22 (1979), no. 3, 271-275.
- [6] C. L. Olsen and G. K. Pedersen, Corona-algebras and their applications to lifting problems, *Math. Scand.* 64 (1989), 63-86.
- [7] C. L. Olsen, A structure theorem for polynomially compact operators, *Amer. J. Math.* 93 (1971), 686-698.
- [8] C. L. Olsen and J. K. Plastiras, Quasialgebraic operators, compact perturbations, and the essential norm, *Michigan Math. J.* 21 (1974), 385-397.
- [9] C. L. Olsen, Norms of compact perturbations of operators, *Pacific J. Math.*, 68, 1 (1977).
- [10] R. Holmes, B. Scranton and J. Ward, Approximation from the space of compact operators and other M-ideals, *Duke Mathematical Journal*, vol. 42, 2 (1975).
- [11] T. Shulman, Lifting of nilpotent contractions, *Bull. London Math. Soc.*, 40, 6 (2008), 1002-1006.
- [12] C.A. Akemann and G.K. Pedersen, Ideal perturbations of elements in C^* -algebras, *Math. Scand.* , 41 (1977), 117-139.
- [13] T. Shulman, Semiprojectivity of universal C^* -algebras generated by algebraic elements, preprint.
- [14] C. K. Chui, D. A. Legg, P. W. Smith, and J. D. Ward, On a question of Olsen concerning compact perturbations of operators, *Michigan Math. J.* 24, 1 (1977), 119-127.
- [15] R. R. Smith and J. D. Ward, A note on polynomial operator approximation, *Proc. Amer. Math. Soc.*, 88, 3 (1983).
- [16] T. Loring and T. Shulman, Noncommutative semialgebraic sets and associated lifting problems, *Transactions of Amer. Math. Soc.*, to appear.
- [17] K. Davidson, Polynomially bounded operators, a survey, in *Operator Algebras and Applications*, NATO ASI Series, Vol. 495, 145-163.
- [18] C. K. Fong and A. R. Sourour, Renorming, similarity and numerical range, *J. London Math. Soc.* 1978; s2-18: 511-518

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEW MEXICO, ALBUQUERQUE, NM 87131, USA.

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN, UNIVERSITETSPARKEN 5, DK-2100 COPENHAGEN, DENMARK